

Projectively related complex Finsler metrics

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Abstract

In this paper we introduce in study the projectively related complex Finsler metrics. We prove the complex versions of the Rapcsák's theorem and characterize the weakly Kähler and generalized Berwald projectively related complex Finsler metrics. The complex version of Hilbert's Fourth Problem is also pointed out. As an application, the projectiveness of a complex Randers metric is described.

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1 Introduction

The problem of projectively related real Finsler metrics is quite old in geometry and its origin is formulated in Hilbert's Fourth Problem: determine the metrics on an open subset in \mathbf{R}^n , whose geodesics are straight lines. Two Finsler metrics, on a common underlying manifold, are called projectively related if any geodesic of the first is also geodesic for the second and the other way around.

The study of projective real Finsler spaces was initiated by L. Berwald, [10, 11], and his studies mainly concern the two dimensional Finsler spaces. Further substantial contributions on this topic are from Rapcsák [23], Misra [20] and, especially, from Z. Szabo [27] and M. Matsumoto [18]. The problem of projective Finsler spaces is strongly connected to projectively related sprays, as Z. Shen pointed out in [26]. The topic of projective real Finsler spaces continues to be of interest for special classes of metrics ([7, 8, 15, 17, 12], etc.).

In complex geometry, T. Aikou studied in [2] the projective flatness of complex Finsler metrics by the projective flatness of Finsler connections.

Part of the general themes from projective real Finsler geometry can be broached in complex Finsler geometry. However, there are meaningful differences comparing to real reasonings, mainly on account of the fact that the Chern-Finsler complex nonlinear connection (the main tool in this geometry), generally does not derive from a spray. Another problem is that in complex Finsler geometry, the notion of complex geodesic curve comports two different nuances, one is in Abate-Patrizio's sense, ([1]), and the second is due to Royden, ([24]). But, these notions don't differ too much. Since a complex geodesic curve in Royden's sense assures that the weakly Kähler condition is satisfied along the curve, we can state that any complex geodesic curve in [24]'s sense is a complex geodesic curve in [1]'s sense.

Our aim in the present paper is to study the projectively related complex Finsler metrics F and \tilde{F} on the complex manifold M , using some ideas from the real case. We have the canonical complex nonlinear connection available, proven to be derived from a complex spray and hence it will become an important tool in our approach. Also, in order to obtain a general characterization of the projectively related complex Finsler metrics we consider the complex geodesics in [1]'s sense.

Subsequently, we have made an overview of the paper's content.

In §2, we recall some preliminary properties of the n - dimensional complex Finsler spaces.

In §3, we introduce the projectively related complex Finsler metrics and then we find some necessary and sufficient condition of projectiveness, (Theorem 3.1 and Corollary 3.1). For two projectively related complex Finsler metrics we show that if one of these is weakly Kähler then, the other must also be weakly Kähler, (Theorem 3.2). We prove some complex versions of the Rapcsák's theorem (Theorems 3.3, 3.4 and 3.5). Next, by means of these theorems we are able to characterize the weakly Kähler and generalized Berwald projectively related complex Finsler metrics, (Theorem 3.6 and Corollary 3.2). Moreover, the complex version of Hilbert's Fourth Problem is emphasized, (Theorem 3.7).

The last part of the paper (§4) is devoted to the projectiveness of the complex Randers metric $\tilde{F} = \alpha + |\beta|$. The necessary and sufficient conditions in which the metrics \tilde{F} and α are projectively related are contained in Theorem 4.3. We prove that the complex Randers metric $\tilde{F} = \alpha + |\beta|$ on a domain D from \mathbf{C}^n is projectively related to the complex Euclidean metric F on D if and only if α is projectively related to the Euclidean metric F and, \tilde{F} is a complex Berwald metric, (Theorem 4.4).

2 Preliminaries

For the beginning we will make a survey of complex Finsler geometry and we will set the basic notions and terminology. For more see [1, 21, 4].

Let M be a n - dimensional complex manifold, $z = (z^k)_{k=\overline{1,n}}$ are complex coordinates in a local chart. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold, and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l} \eta^l$.

A *complex Finsler space* is a pair (M, F) , where $F : T'M \rightarrow \mathbb{R}^+$ is a continuous function satisfying the conditions:

- i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;
- ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
- iii) $F(z, \lambda\eta) = |\lambda|F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;
- iv) the Hermitian matrix $(g_{i\bar{j}}(z, \eta))$ is positive definite, where $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, it means that the indicatrix is strongly pseudo-convex.

We say that a function f on $T'M$ is (p, q) - *homogeneous* with respect to η iff $f(z, \lambda\eta) = \lambda^p \bar{\lambda}^q f(z, \eta)$, for any $\lambda \in \mathbb{C}$. For instance, $L := F^2$ is a $(1, 1)$ - homogeneous function.

Roughly speaking, the geometry of a complex Finsler space consists of the study of geometric objects of the complex manifold $T'M$ endowed with the Hermitian metric structure defined by $g_{i\bar{j}}$. Thus, the first step is to study sections of the complexified tangent bundle of $T'M$, which is decomposed into the sum $T_C(T'M) = T'(T'M) \oplus T''(T'M)$.

Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ be its conjugate. At this point, the idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of this geometry. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the (*c.n.c.*) which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k} \delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k} \dot{\partial}'_j$. By conjugation everywhere we have obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Certainly, a main problem in this geometry is to determine a (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M, F) .

The next step is the action of a derivative law D on the sections of $T_C(T'M)$. A Hermitian connection D , of $(1, 0)$ - type, which satisfies in addi-

tion $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, is called Chern-Finsler connection (cf. [1]). Locally, it is given by the following coefficients (cf. [21]):

$$N_j^i := g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l = L_{lj}^i \eta^l ; L_{jk}^i := g^{\bar{l}i} \delta_k g_{j\bar{l}} ; C_{jk}^i := g^{\bar{l}i} \dot{\partial}_k g_{j\bar{l}}, \quad (2.1)$$

where here and further on δ_k is related to the Chern-Finsler (*c.n.c.*) and $D_{\delta_k} \delta_j = L_{jk}^i \delta_i$, $D_{\dot{\partial}_k} \dot{\partial}_j = C_{jk}^i \dot{\partial}_i$.

Let us recall that in [1]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* iff $T_{jk}^i = 0$, *Kähler* iff $T_{jk}^i \eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}} T_{jk}^i \eta^j \bar{\eta}^l = 0$, where $T_{jk}^i := L_{jk}^i - L_{kj}^i$. In [13] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M , so-called *purely Hermitian metrics* in [21], i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$, all these kinds of Kähler coincide.

The Chern-Finsler (*c.n.c.*) generally, does not derive from a spray, but it always determine a complex spray with the local coefficients $G^i = \frac{1}{2} N_j^i \eta^j$. Instead, G^i induce a (*c.n.c.*) by $\overset{c}{N}_j^i := \dot{\partial}_j G^i$ called *canonical* in [21], where it is proved that it coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler. Note that $2G^i = N_j^i \eta^j = \overset{c}{N}_j^i \eta^j$, and so $\eta^k \overset{c}{\delta}_k = \eta^k \delta_k$, where $\overset{c}{\delta}_k$ is related to canonical (*c.n.c.*), i.e. $\overset{c}{\delta}_k := \frac{\partial}{\partial z^k} - \overset{c}{N}_k^j \dot{\partial}_j$. Additionally, in the Kähler case, we have $\overset{c}{\delta}_k = \delta_k$.

In [4] we have proven that the complex Finsler space (M, F) is *generalized Berwald* iff $\dot{\partial}_h G^i = 0$ and (M, F) is a complex *Berwald* space iff it is Kähler and generalized Berwald.

3 Projectively related complex Finsler metrics

In Abate-Patrizio's sense, ([1] p. 101), a complex geodesic curve is given by $D_{T^h + \overline{T^h}} T^h = \theta^*(T^h, \overline{T^h})$, where $\theta^* = g^{\bar{m}k} g_{i\bar{p}} (L_{j\bar{m}}^{\bar{p}} - L_{\bar{m}\bar{j}}^{\bar{p}}) dz^i \wedge d\bar{z}^j \otimes \delta_k$, for which it is proven in [21] that $\theta^{*k} = 2g^{\bar{j}k} \overset{c}{\delta}_{\bar{j}} L$ and θ^{*i} is vanishing if and only if the space is weakly Kähler. Thus, the equations of a complex geodesic $z = z(s)$ of (M, L) , with s a real parameter, in [1]'s sense can be rewritten as

$$\frac{d^2 z^i}{ds^2} + 2G^k(z(s), \frac{dz}{ds}) = \theta^{*i}(z(s), \frac{dz}{ds}) ; i = \overline{1, n}, \quad (3.1)$$

where by $z^i(s)$, $i = \overline{1, n}$, we denote the coordinates along of curve $z = z(s)$.

We note that the functions θ^{*i} are $(1, 1)$ - homogeneous with respect to η , i.e. $(\dot{\partial}_k \theta^{*i}) \eta^k = \theta^{*i}$ and $(\dot{\partial}_{\bar{k}} \theta^{*i}) \bar{\eta}^k = \theta^{*i}$.

Let \tilde{L} be another complex Finsler metric on the underlying manifold M .

Definition 3.1. *The complex Finsler metrics L and \tilde{L} on the manifold M , are called projectively related if they have the same complex geodesics as point sets.*

This means that for any complex geodesic $z = z(s)$ of (M, L) there is a transformation of its parameter s , $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesic of (M, \tilde{L}) and, conversely.

We suppose that $z = z(s)$ is a complex geodesic of (M, L) . Thus, it satisfies (3.1). Taking an arbitrary transformation of the parameter $t = t(s)$, with $\frac{dt}{ds} > 0$, the equations (3.1) cannot in general be preserved. Indeed, for the new parameter t we have

$$\frac{dz^i}{ds} = \frac{dz^i}{dt} \frac{dt}{ds} ; \quad \frac{d^2 z^i}{ds^2} = \frac{d^2 z^i}{dt^2} \left(\frac{dt}{ds} \right)^2 + \frac{dz^i}{dt} \frac{d^2 t}{ds^2} ; \quad \theta^{*k} \left(z, \frac{dz}{ds} \right) = \left(\frac{dt}{ds} \right)^2 \theta^{*k} \left(z, \frac{dz}{dt} \right).$$

Then,

$$\left[\frac{d^2 z^i}{dt^2} + 2G^i \left(z, \frac{dz}{dt} \right) - \theta^{*i} \left(z, \frac{dz}{dt} \right) \right] \left(\frac{dt}{ds} \right)^2 = \frac{d^2 z^i}{ds^2} - \frac{dz^i}{dt} \frac{d^2 t}{ds^2} + 2G^i \left(z, \frac{dz}{ds} \right) - \theta^{*i} \left(z, \frac{dz}{ds} \right) = - \frac{dz^i}{dt} \frac{d^2 t}{ds^2}.$$

Therefore, the equations (3.1) in parameter t are

$$\frac{d^2 z^i}{dt^2} + 2G^i \left(z(t), \frac{dz}{dt} \right) = \theta^{*i} \left(z(t), \frac{dz}{dt} \right) - \frac{dz^i}{dt} \frac{d^2 t}{ds^2} \frac{1}{\left(\frac{dt}{ds} \right)^2} ; \quad i = \overline{1, n}, \quad (3.2)$$

which is equivalent to

$$\frac{\frac{d^2 z^i}{dt^2} + 2G^i \left(z, \frac{dz}{dt} \right) - \theta^{*i} \left(z, \frac{dz}{dt} \right)}{\frac{dz^i}{dt}} = - \frac{d^2 t}{ds^2} \frac{1}{\left(\frac{dt}{ds} \right)^2} ; \quad i = \overline{1, n}. \quad (3.3)$$

We can rewrite (3.3), taking for i two different values, as

$$\frac{\frac{d^2 z^i}{dt^2} + 2G^j \left(z, \frac{dz}{dt} \right) - \theta^{*j} \left(z, \frac{dz}{dt} \right)}{\frac{dz^j}{dt}} = \frac{\frac{d^2 z^k}{dt^2} + 2G^k \left(z, \frac{dz}{dt} \right) - \theta^{*k} \left(z, \frac{dz}{dt} \right)}{\frac{dz^k}{dt}} = - \frac{d^2 t}{ds^2} \frac{1}{\left(\frac{dt}{ds} \right)^2}, \quad (3.4)$$

for any $j, k = \overline{1, n}$.

Corresponding to the complex Finsler metric \tilde{L} on the same manifold M , we have the spray coefficients \tilde{G}^i and the functions $\tilde{\theta}^{*i}$. If L and \tilde{L} are projectively related, then $z = z(\tilde{s})$ is a complex geodesic of (M, \tilde{L}) , where \tilde{s} is

the parameter with respect to \tilde{L} . Now, we assume that the same parameter t is transformed by $t = t(\tilde{s})$ and as above we obtain

$$\frac{\frac{d^2 x^i}{dt^2} + 2\tilde{G}^i(z, \frac{dz}{dt}) - \tilde{\theta}^{*i}(z, \frac{dz}{dt})}{\frac{dz^i}{dt}} = -\frac{d^2 t}{d\tilde{s}^2} \frac{1}{\left(\frac{dt}{d\tilde{s}}\right)^2}; \quad i = \overline{1, n}. \quad (3.5)$$

The difference between (3.3) and (3.5) gives

$$2\tilde{G}^i(z, \frac{dz}{dt}) - \tilde{\theta}^{*i}(z, \frac{dz}{dt}) = 2G^i(z, \frac{dz}{dt}) - \theta^{*i}(z, \frac{dz}{dt}) + \left[\frac{d^2 t}{ds^2} \frac{1}{\left(\frac{dt}{ds}\right)^2} - \frac{d^2 t}{d\tilde{s}^2} \frac{1}{\left(\frac{dt}{d\tilde{s}}\right)^2} \right] \frac{dz^i}{dt}. \quad (3.6)$$

On the geodesic curves, it can be rewritten more generally as

$$2\tilde{G}^i(z, \frac{dz}{dt}) - \tilde{\theta}^{*i}(z, \frac{dz}{dt}) = 2G^i(z, \frac{dz}{dt}) - \theta^{*i}(z, \frac{dz}{dt}) + 2P(z, \frac{dz}{dt}) \frac{dz^i}{dt}, \quad (3.7)$$

for any $i = \overline{1, n}$, where P is a smooth function on $T'M$ with complex values.

Denoting by $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$, the homogeneity properties of the functions $\tilde{\theta}^{*i}$ and θ^{*i} give $(\dot{\partial}_k B^i)\eta^k = B^i$ and $(\dot{\partial}_{\bar{k}} B^i)\bar{\eta}^k = B^i$. Moreover the relations (3.7) become

$$\tilde{G}^i = G^i + B^i + P\eta^i. \quad (3.8)$$

Now, we use their homogeneity properties, going from η to $\lambda\eta$. Thus, differentiating in (3.8) with respect to η and $\bar{\eta}$ and then setting $\lambda = 1$, we obtain

$$B^i = [(\dot{\partial}_k P)\eta^k - P]\eta^i \quad \text{and} \quad B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i \quad (3.9)$$

and so,

$$(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P, \quad (3.10)$$

for any $i = \overline{1, n}$.

Lemma 3.1. *Between the spray coefficients \tilde{G}^i and G^i of the metrics L and \tilde{L} on the manifold M there are the relations $\tilde{G}^i = G^i + B^i + P\eta^i$, for any $i = \overline{1, n}$, where P is a smooth function on $T'M$ with complex values, if and only if $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k \eta^i$, $B^i(z, \eta) = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i$, for any $i = \overline{1, n}$, and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$.*

From above considerations we obtain

Lemma 3.2. *If the complex Finsler metrics L and \tilde{L} on the manifold M are projectively related, then there is a smooth function P on $T'M$ with complex values, satisfying $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$, such that*

$$\tilde{G}^i(z, \eta) = G^i(z, \eta) + (\dot{\partial}_k P)\eta^k \eta^i \quad \text{and} \quad B^i(z, \eta) = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i; \quad i = \overline{1, n}. \quad (3.11)$$

Remark 3.1. We denote $S := (\dot{\partial}_k P)\eta^k$ and $Q := -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k$. The $(2, 0)$ -homogeneity with respect to η of the functions \tilde{G}^i and G^i implies the $(1, 0)$ -homogeneity of S , and the $(1, 1)$ - homogeneity of B^i gives that Q is $(0, 1)$ -homogeneous.

Conversely, under assumption that $z = z(s)$ is a complex geodesic of (M, L) , we show that the complex Finsler metric \tilde{L} with the spray coefficients \tilde{G}^i given by

$$\tilde{G}^i = G^i + B^i + P\eta^i,$$

where P is a smooth function on $T'M$ with complex values, is projectively related to L , i.e. there is a parametrization $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesic of (M, \tilde{L}) .

If there is a parametrization $\tilde{s} = \tilde{s}(s)$ then we have

$$\frac{d^2 z^i}{d\tilde{s}^2} = -2G^i(z, \frac{dz}{d\tilde{s}}) + \theta^{*i}(z, \frac{dz}{d\tilde{s}}) - \frac{d^2 \tilde{s}}{ds^2} \frac{1}{(\frac{d\tilde{s}}{ds})^2} \frac{dz^i}{d\tilde{s}}, \text{ for any } i = \overline{1, n}.$$

Now, using (3.11), it results

$$\frac{d^2 z^i}{d\tilde{s}^2} = -2\tilde{G}^i(z, \frac{dz}{d\tilde{s}}) + \tilde{\theta}^{*i}(z, \frac{dz}{d\tilde{s}}) + \left(2P(z, \frac{dz}{d\tilde{s}}) - \frac{d^2 \tilde{s}}{ds^2} \frac{1}{(\frac{d\tilde{s}}{ds})^2} \right) \frac{dz^i}{d\tilde{s}}; \quad i = \overline{1, n}.$$

So, $z = z(\tilde{s}(s))$ is a geodesic of (M, \tilde{L}) if and only if

$$\left(2P(z, \frac{dz}{d\tilde{s}}) - \frac{d^2 \tilde{s}}{ds^2} \frac{1}{(\frac{d\tilde{s}}{ds})^2} \right) \frac{dz^i}{d\tilde{s}} = 0; \quad i = \overline{1, n}. \quad (3.12)$$

Supposing the complex geodesic curve is not a line, it results

$$2P(z, \frac{dz}{d\tilde{s}}) \frac{d\tilde{s}}{ds} = \frac{d^2 \tilde{s}}{ds^2}. \quad (3.13)$$

Denoting by $u(s) := \frac{d\tilde{s}}{ds}$, we have $\frac{d^2 \tilde{s}}{ds^2} = \frac{du}{ds}$ and so, $2P(z, \frac{dz}{d\tilde{s}})u = \frac{du}{ds}$. We obtain $u = ae^{\int 2P(z, \frac{dz}{d\tilde{s}})ds}$. From here, it results that there is

$$\tilde{s}(s) = a \int e^{\int 2P(z, \frac{dz}{d\tilde{s}})ds} ds + b,$$

where a, b are arbitrary constants.

Corroborating all above results we have proven.

Theorem 3.1. Let L and \tilde{L} be complex Finsler metrics on the manifold M . Then L and \tilde{L} are projectively related if and only if there is a smooth function P on $T'M$ with complex values, such that

$$\tilde{G}^i = G^i + B^i + P\eta^i; \quad i = \overline{1, n}. \quad (3.14)$$

As a consequence of Lemma 3.1 we have the following.

Corollary 3.1. *Let L and \tilde{L} be the complex Finsler metrics on the manifold M . L and \tilde{L} are projectively related if and only if there is a smooth function P on $T'M$ with complex values, such that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k \eta^i$, $B^i(z, \eta) = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i$, for any $i = \overline{1, n}$, and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$.*

The relations (3.14) between the spray coefficients \tilde{G}^i and G^i of the projectively related complex Finsler metrics L and \tilde{L} will be called *projective change*.

Theorem 3.2. *Let L and \tilde{L} be two complex Finsler metrics on the manifold M , which are projectively related. Then, L is weakly Kähler if and only if \tilde{L} is also weakly Kähler. In this case, the projective change is $\tilde{G}^i = G^i + P\eta^i$, where P is a $(1, 0)$ - homogeneous function.*

Proof. We assume that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k \eta^i$, $B^i = \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i}) = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$.

If L is weakly Kähler then $\theta^{*i} = 0$ and so, $\tilde{\theta}^{*i} = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i$ which contracted by $\tilde{g}_{i\bar{r}}\bar{\eta}^r = \dot{\partial}_l \tilde{L}$, gives $\tilde{\theta}^{*i}\tilde{g}_{i\bar{r}}\bar{\eta}^r = -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \tilde{L}$.

But, $\tilde{\theta}^{*i}\tilde{g}_{i\bar{r}}\bar{\eta}^r = 0$. Thus, $(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = 0$, which implies $\tilde{\theta}^{*i} = 0$, i.e. \tilde{L} is weakly Kähler and $P = (\dot{\partial}_k P)\eta^k$. So that, we obtain $\tilde{G}^i = G^i + P\eta^i$. The converse implication results immediately by the same way. \square

Lemma 3.3. *Let (M, L) be a complex Finsler space and \tilde{L} a complex Finsler metric on M . The spray coefficients \tilde{G}^i and G^i of the metrics L and \tilde{L} satisfy*

$$\tilde{G}^i = G^i + \frac{1}{2}\tilde{g}^{\bar{r}i} \left(\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) \right); \quad i = \overline{1, n}. \quad (3.15)$$

Proof. Having $\delta_k^c \tilde{L} = \frac{\partial \tilde{L}}{\partial z^k} - N_k^l (\dot{\partial}_l \tilde{L})$, by a direct computation we obtain

$$\dot{\partial}_{\bar{r}}(\delta_k^c \tilde{L}) = \dot{\partial}_{\bar{r}} \left(\frac{\partial \tilde{L}}{\partial z^k} - N_k^l (\dot{\partial}_l \tilde{L}) \right) = \frac{\partial^2 \tilde{L}}{\partial z^k \partial \bar{\eta}^r} - (\dot{\partial}_{\bar{r}} N_k^l)(\dot{\partial}_l \tilde{L}) - N_k^l \tilde{g}_{l\bar{r}}, \text{ which}$$

contracted with $\tilde{g}^{\bar{r}i}\eta^k$, and taking into account $\eta^k \delta_k^c = \eta^k \delta_k$, implies that

$$\tilde{g}^{\bar{r}i} \dot{\partial}_{\bar{r}}(\delta_k^c \tilde{L})\eta^k = \tilde{g}^{\bar{r}i} \dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k = 2\tilde{G}^i - 2\tilde{g}^{\bar{r}i}(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) - 2G^i \text{ and so (3.15) is justified. } \square$$

Next, we prove some complex versions of the Rapcsák's theorem.

Theorem 3.3. *Let L and \tilde{L} be complex Finsler metrics on the manifold M . Then, L and \tilde{L} are projectively related if and only if*

$$\frac{1}{2} \left(\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) \right) = P(\dot{\partial}_{\bar{r}} \tilde{L}) + B^i \tilde{g}_{i\bar{r}}; \quad r = \overline{1, n}, \quad (3.16)$$

$$\text{with } P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})].$$

Proof. We assume that L and \tilde{L} are projectively related. Then, by Theorem 3.1 and (3.15) we have

$$B^i + P\eta^i = \frac{1}{2}\tilde{g}^{\bar{r}i} \left(\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L}) \right); \quad i = \overline{1, n}. \quad (3.17)$$

First, if these relations are contracted by $\tilde{g}_{i\bar{m}}\bar{\eta}^m$, we get

$$-\frac{1}{2}\theta^{*i}(\dot{\partial}_i \tilde{L}) + P\tilde{L} = \frac{1}{2}\dot{\partial}_{\bar{m}}(\delta_k \tilde{L})\eta^k\bar{\eta}^m + (\dot{\partial}_{\bar{m}}G^l)\bar{\eta}^m(\dot{\partial}_l \tilde{L}),$$

because $B^i\tilde{g}_{i\bar{m}}\bar{\eta}^m = -\frac{1}{2}\theta^{*i}(\dot{\partial}_i \tilde{L})$. But, the $(2, 0)$ -homogeneity of the functions G^i leads to $(\dot{\partial}_{\bar{m}}G^l)\bar{\eta}^m = 0$ and $\dot{\partial}_{\bar{m}}(\delta_k \tilde{L})\eta^k\bar{\eta}^m = (\delta_k \tilde{L})\eta^k$. Thus, $P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]$. Second, contracting into (3.17) only by $\tilde{g}_{i\bar{m}}$, we obtain (3.16).

Conversely, plugging the formulas (3.16) into (3.15), it results (3.14) with $P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]$, i.e. L and \tilde{L} are projectively related. \square

Theorem 3.4. *Let L and \tilde{L} be the complex Finsler metrics on the manifold M . Then, L and \tilde{L} are projectively related if and only if*

$$\begin{aligned} \dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L}) &= \frac{1}{\tilde{L}}(\delta_k \tilde{L})\eta^k(\dot{\partial}_{\bar{r}} \tilde{L}); \\ B^r &= -\frac{1}{2\tilde{L}}\theta^{*l}(\dot{\partial}_l \tilde{L})\eta^r; \quad r = \overline{1, n}; \\ P &= \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]. \end{aligned} \quad (3.18)$$

Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k\eta^i$.

Proof. By Corollary 3.1, if L and \tilde{L} are projectively related, then there is a smooth function P on $T'M$ with complex values, such that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$, for any $i = \overline{1, n}$, and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$. Using (3.15) it results

$$(\dot{\partial}_k P)\eta^k\eta^i = \frac{1}{2}\tilde{g}^{\bar{r}i} \left(\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L}) \right); \quad i = \overline{1, n}, \quad (3.19)$$

which contracted firstly by $\tilde{g}_{i\bar{m}}$ and secondly by $\tilde{g}_{i\bar{m}}\bar{\eta}^m$ give

$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L}) = 2(\dot{\partial}_k P)\eta^k(\dot{\partial}_{\bar{r}} \tilde{L})$ and $(\dot{\partial}_k P)\eta^k = \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k$ respectively, since $\delta_k \tilde{L}$ is $(1, 1)$ -homogeneous.

Now, contracting $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$ with $\tilde{g}_{i\bar{m}}\bar{\eta}^m$, it leads to $(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = \frac{1}{2\tilde{L}}\theta^{*i}(\dot{\partial}_i \tilde{L})$, because $B^i\tilde{g}_{i\bar{m}}\bar{\eta}^m = -\frac{1}{2}\theta^{*i}(\dot{\partial}_i \tilde{L})$. Adding the last two relations obtained, it results $P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]$.

Conversely, replacing the first condition of (3.18) into (3.15) we obtain $\tilde{G}^i = G^i + S\eta^i$, where $S := \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k$.

Now, having $P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]$, we obtain

$$(\dot{\partial}_k P)\eta^k = \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k = S \text{ and } (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = \frac{1}{2\tilde{L}}\theta^{*i}(\dot{\partial}_i \tilde{L}).$$

Thus, these lead to $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k \eta^i$, $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$. \square

Plugging $\tilde{L} = \tilde{F}^2$ into (3.18) we have proven another equivalent complex version of Rapcsák's theorem.

Theorem 3.5. *Let F and \tilde{F} be the complex Finsler metrics on the manifold M . Then, F and \tilde{F} are projectively related if and only if*

$$\begin{aligned} \dot{\partial}_{\bar{r}}(\delta_k \tilde{F})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{F}) &= \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k(\dot{\partial}_{\bar{r}} \tilde{F}); \\ B^r &= -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l \tilde{F})\eta^r; \quad r = \overline{1, n}; \\ P &= \frac{1}{\tilde{F}}[(\delta_k \tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{F})]. \end{aligned} \quad (3.20)$$

Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k \eta^i$.

Theorem 3.6. *Let L be a weakly Kähler complex Finsler metric on the manifold M and \tilde{L} another complex Finsler metric on M . Then, L and \tilde{L} are projectively related if and only if \tilde{L} is weakly Kähler and*

$$\begin{aligned} \dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) &= 2P(\dot{\partial}_{\bar{r}} \tilde{L}); \quad r = \overline{1, n}, \\ P &= \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k. \end{aligned} \quad (3.21)$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ and P is $(1, 0)$ - homogeneous.

Proof. Having in mind the Theorems 3.2 and 3.4 the direct implication is obvious. For the converse, we have $B^i = \theta^{*i} = \tilde{\theta}^{*i} = 0$, because L and \tilde{L} are weakly Kähler, which together with (3.21) are sufficient conditions for the projectiveness of the metrics L and \tilde{L} . Now, plugging (3.21) into (3.15) it results $\tilde{G}^i = G^i + P\eta^i$ and the $(1, 0)$ -homogeneity of P . \square

Let us pay more attention to Theorem 3.5. As its consequence, we have.

Corollary 3.2. *Let F be a generalized Berwald metric on the manifold M and \tilde{F} another complex Finsler metric on M . Then, F and \tilde{F} are projective if and only if*

$$\begin{aligned}\dot{\partial}_{\bar{r}}(\delta_k \tilde{F})\eta^k &= \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k(\dot{\partial}_{\bar{r}}\tilde{F}) ; B^r = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l \tilde{F})\eta^r ; \\ P &= \frac{1}{\tilde{F}}[(\delta_k \tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{F})],\end{aligned}\quad (3.22)$$

for any $r = \overline{1, n}$. Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k \eta^i$ and \tilde{F} is also generalized Berwald.

Proof. The equivalence results by Theorem 3.5 in which $\dot{\partial}_{\bar{r}}G^l = 0$, because F is a generalized Berwald metric. In order to show that \tilde{F} is generalized Berwald, we compute

$\dot{\partial}_{\bar{r}}[\frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k] = -\frac{1}{\tilde{F}^2}(\dot{\partial}_{\bar{r}}\tilde{F})(\delta_k \tilde{F})\eta^k + \frac{1}{\tilde{F}}\dot{\partial}_{\bar{r}}(\delta_k \tilde{F})\eta^k = 0$, by using the first identity from (3.22). Now, differentiating the projective change $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k \eta^i$ with respect to $\bar{\eta}^r$ it results $\dot{\partial}_{\bar{r}}\tilde{G}^l = 0$, i.e. \tilde{F} is generalized Berwald. \square

In particular, if F is a Kähler metric, then Theorem 3.6 and Corollary 3.2 imply

Corollary 3.3. *Let F be a complex Berwald metric on the manifold M and \tilde{F} another complex Finsler metric on M . Then, F and \tilde{F} are projectively related if and only if \tilde{F} is weakly Kähler and*

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{F})\eta^k = P(\dot{\partial}_{\bar{r}}\tilde{F}) ; r = \overline{1, n} ; P = \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k. \quad (3.23)$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ and \tilde{F} is generalized Berwald.

Proposition 3.1. *Let F and \tilde{F} be two projectively related complex Finsler metrics on the manifold M . If P is $(1, 0)$ - homogeneous with respect to η and F is generalized Berwald, then P is holomorphic with respect to η .*

Proof. We have $\tilde{G}^i = G^i + B^i + P\eta^i$, with P homogenous of $(1, 0)$ - degree. This implies $B^i = 0$ and so, by Corollary 3.2, $\theta^{*l}(\dot{\partial}_l \tilde{F}) = 0$. So that, $P = \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k \eta^i$ and, it has the property $\dot{\partial}_{\bar{r}}P = 0$. \square

Proposition 3.2. *Let F and \tilde{F} be two projectively related complex Finsler metrics on the manifold M . If P is $(0, 1)$ - homogeneous with respect to η and F is generalized Berwald, then $B^i = -P\eta^i$, for any $i = \overline{1, n}$, and the projective change is $\tilde{G}^i = G^i$.*

Proof. Let be the projective change $\tilde{G}^i = G^i + B^i + P\eta^i$, with P homogenous of $(0, 1)$ - degree. Since \tilde{G}^i and G^i are $(2, 0)$ - homogeneous and $B^i, P\eta^i$ are $(1, 1)$ - homogeneous, it follows $\tilde{G}^i = G^i$ and $B^i = -P\eta^i$. \square

Further on, the complex version of the *Hilbert's Fourth Problem* is approached.

Theorem 3.7. *Let L be complex Euclidean metric on a domain D from \mathbf{C}^n and \tilde{L} another complex Finsler metric on D . Then, L and \tilde{L} are projectively related if and only if \tilde{L} is weakly Kähler and*

$$\tilde{G}^i = \frac{1}{2\tilde{L}} \frac{\partial \tilde{L}}{\partial z^k} \eta^k \eta^i ; i = \overline{1, n} . \quad (3.24)$$

Moreover, \tilde{L} is generalized Berwald.

Proof. The complex Euclidean metric $L := |\eta|^2 = \sum_{k=1}^n \eta^k \bar{\eta}^k$ is Kähler with the local spray coefficients $G^i = 0$, for any $i = \overline{1, n}$. By these assumptions, the conditions (3.21) can be rewritten as

$$\dot{\partial}_{\bar{r}} \left(\frac{\partial \tilde{L}}{\partial z^k} \right) \eta^k = 2P(\dot{\partial}_{\bar{r}} \tilde{L}), \quad (3.25)$$

for any $r = \overline{1, n}$, where $P = \frac{1}{2\tilde{L}} \frac{\partial \tilde{L}}{\partial z^k} \eta^k$. Further on, by contraction in (3.25) with $\tilde{g}^{\bar{r}i}$ and since $\tilde{G}^i = \frac{1}{2} \tilde{g}^{\bar{r}i} \dot{\partial}_{\bar{r}} \left(\frac{\partial \tilde{L}}{\partial z^k} \right) \eta^k$, using again (3.25) it follows that $\tilde{G}^i = P\eta^i$ which is (3.24). The converse is obvious. \square

Taking $\tilde{L} = \tilde{F}^2$ into (3.24), it becomes

$$\tilde{G}^i = \frac{1}{\tilde{F}} \frac{\partial \tilde{F}}{\partial z^k} \eta^k \eta^i ; i = \overline{1, n} . \quad (3.26)$$

Some examples of complex Finsler metrics which are projectively related to the complex Euclidean metric are given by the following purely Hermitian metrics defined over the disk $\Delta_r^n = \left\{ z \in \mathbf{C}^n, |z| < r, r := \sqrt{\frac{1}{|\varepsilon|}} \right\} \subset \mathbf{C}^n$:

$$\tilde{L}(z, \eta) := \frac{|\eta|^2 + \varepsilon (|z|^2 |\eta|^2 - |\langle z, \eta \rangle|^2)}{(1 + \varepsilon |z|^2)^2} ; \varepsilon < 0, \quad (3.27)$$

where $|z|^2 := \sum_{k=1}^n z^k \bar{z}^k$, $\langle z, \eta \rangle := \sum_{k=1}^n z^k \bar{\eta}^k$ and $|\langle z, \eta \rangle|^2 = \langle z, \eta \rangle \overline{\langle z, \eta \rangle}$. They are Kähler and in particular, for $\varepsilon = -1$ we obtain the *Bergman metric* on the unit disk $\Delta^n := \Delta_1^n$. Their geodesics are segments of straight lines.

4 Projectiveness of a complex Randers metric

We consider $\beta(z, \eta) := b_i(z)\eta^i$ a differential $(1, 0)$ -form and $\alpha(z, \eta) := \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ a purely Hermitian metric on the manifold M . By these objects we have defined the complex Randers metric $\tilde{F} = \alpha + |\beta|$ on $T'M$ with

$$\begin{aligned}\frac{\partial \alpha}{\partial \eta^i} &= \frac{1}{2\alpha} l_i \ ; \ \frac{\partial |\beta|}{\partial \eta^i} = \frac{\bar{\beta}}{2|\beta|} b_i \ ; \ \tilde{\eta}_i := \frac{\partial \tilde{L}}{\partial \eta^i} = \frac{\tilde{F}}{\alpha} l_i + \frac{\tilde{F}\bar{\beta}}{|\beta|} b_i, \\ \tilde{G}^i &= \overset{a}{G}^i + \frac{1}{2\gamma} \left(l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} - \frac{\beta^2}{|\beta|^2} \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r \right) \xi^i \eta^j + \frac{\beta}{4|\beta|} k^{\bar{r}i} \frac{\partial b_{\bar{r}}}{\partial z^j} \eta^j \\ l_i &:= a_{i\bar{j}} \bar{\eta}^j \ ; \ b^i := a^{\bar{j}i} b_{\bar{j}} \ ; \ ||b||^2 := a^{\bar{j}i} b_i b_{\bar{j}} \ ; \ b^{\bar{i}} := \bar{b}^i,\end{aligned}$$

where $\overset{a}{G}^i = \frac{1}{2} \overset{a}{N}_j^i \eta^j$ are the spray coefficients of the purely Hermitian metric α and $\gamma := \tilde{L} + \alpha^2(||b||^2 - 1)$, $\xi^i := \bar{\beta}\eta^i + \alpha^2 b^i$, $k^{\bar{r}i} := 2\alpha a^{\bar{j}i} + \frac{2(\alpha||b||^2 + 2|\beta|)}{\gamma} \eta^i \bar{\eta}^r - \frac{2\alpha^3 b^i \bar{b}^r}{\gamma} - \frac{2\alpha}{\gamma} (\bar{\beta}\eta^i \bar{b}^r + \beta b^i \bar{\eta}^r)$.

Moreover, the complex Randers metric \tilde{F} is weakly Kähler if and only if

$$\begin{aligned}\frac{\alpha^2 |\beta|}{\gamma \delta} \left[\beta \frac{\alpha ||b||^2 + |\beta|}{|\beta|} \frac{\partial b_{\bar{m}}}{\partial z^r} \bar{\eta}^m + \bar{\beta} \left(\frac{\partial b_r}{\partial z^l} - b^m \frac{\partial a_{l\bar{m}}}{\partial z^r} \right) \eta^l - \alpha |\beta| b^m \frac{\partial b_{\bar{m}}}{\partial z^r} \right] \eta^r C_k \\ - \left(\alpha \bar{\beta} F_{kl} + \alpha b_l \frac{\partial b_{\bar{r}}}{\partial z^k} \bar{\eta}^r + 2|\beta| a_{l\bar{r}} \Gamma_{jk}^{\bar{r}} \bar{\eta}^j \right) \eta^l + \alpha b_k \frac{\partial b_{\bar{m}}}{\partial z^r} \bar{\eta}^m \eta^r = 0,\end{aligned}\quad (4.1)$$

where $C_j := \delta \left(\frac{1}{\alpha^2} l_j - \frac{\bar{\beta}}{|\beta|^2} b_j \right)$, with $\delta := \frac{\alpha^2 ||b||^2 - |\beta|^2}{2\gamma} - \frac{n|\beta|}{2F}$, $F_{il} := \frac{\partial b_l}{\partial z^i} - \frac{\partial b_i}{\partial z^l}$ and $\Gamma_{ji}^{\bar{r}} := \frac{1}{2} a^{\bar{r}k} \left(\frac{\partial a_{k\bar{j}}}{\partial z^i} - \frac{\partial a_{i\bar{j}}}{\partial z^k} \right)$. For more details see [3].

Theorem 4.1. ([4]) *Let (M, \tilde{F}) be a connected complex Randers space. Then, it is a generalized Berwald space if and only if $(\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r) \eta^j = 0$.*

Theorem 4.2. ([4]) *Let (M, \tilde{F}) be a connected complex Randers space. Then, it is a complex Berwald space if and only if it is both generalized Berwald and weakly Kähler. Moreover, α is Kähler and $\tilde{N}_j^i = \overset{a}{N}_j^i$.*

First, our aim is to determine the necessary and sufficient conditions in which the complex Randers metric \tilde{F} is projectively related to the Hermitian metric α . A simple computation shows that,

$$(\delta_k \tilde{F}) \eta^k = (\delta_k |\beta|) \eta^k = \frac{1}{2|\beta|} (\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^k} + \beta \frac{\partial b_{\bar{r}}}{\partial z^k} \bar{\eta}^r) \eta^k, \quad (4.2)$$

because $(\delta_k \alpha) \eta^k = 0$ and

$$\theta^{*i}(\dot{\partial}_i \tilde{F}) = -\frac{\bar{\beta}}{2|\beta|} \Gamma_{i\bar{j}}^k b_k \eta^i \bar{\eta}^j. \quad (4.3)$$

Taking into account Theorem 4.1 we have proven

Lemma 4.1. *Let (M, \tilde{F}) be a connected complex Randers space. Then, (M, \tilde{F}) is a generalized Berwald space if and only if $(\delta_k |\beta|) \eta^k = 0$.*

Theorem 4.3. *Let (M, \tilde{F}) be a connected complex Randers space. Then,*
i) α and \tilde{F} are projectively related if and only if \tilde{F} is generalized Berwald and $B^i = -P\eta^i$, for any $i = \overline{1, n}$, where $P = -\frac{\bar{\beta}}{2\tilde{F}|\beta|} \Gamma_{i\bar{j}}^k b_k \eta^i \bar{\eta}^j$;

ii) α is Kähler and α is projectively related to \tilde{F} if and only if \tilde{F} is a complex Berwald metric.

In these cases, the projective change is $\tilde{G}^i = \overset{a}{G}^i$.

Proof. We first prove i). The purely Hermitian property of the metric α implies that it is generalized Berwald. If α and \tilde{F} are projectively related, then by Corollary 3.2 it results that \tilde{F} is also generalized Berwald. So that, by (4.2), (4.3) and Lemma 4.1, the conditions (3.22) reduce to $B^i = -P\eta^i$, for any $i = \overline{1, n}$, where $P = -\frac{\bar{\beta}}{2\tilde{F}|\beta|} \Gamma_{i\bar{j}}^k b_k \eta^i \bar{\eta}^j$.

Conversely, if \tilde{F} is generalized Berwald, then the first condition from (3.22) is identically satisfied and by (4.3), $B^i = -\frac{1}{\tilde{F}} \theta^{*l}(\dot{\partial}_l \tilde{F}) \eta^i$ and $P = \frac{1}{\tilde{F}} \theta^{*i}(\dot{\partial}_i \tilde{F})$. All these conditions imply the projectiveness of the metrics α and \tilde{F} . ii) is a consequence of i), under assumptions of Kähler for the metrics α and \tilde{F} , respectively. \square

Example. Let $\Delta = \{(z, w) \in \mathbf{C}^2, |w| < |z| < 1\}$ be the Hartogs triangle with the Kähler-purely Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} (\log \frac{1}{(1 - |z|^2)(|z|^2 - |w|^2)}); \quad \alpha^2(z, w; \eta, \theta) = a_{i\bar{j}} \eta^i \bar{\eta}^j, \quad (4.4)$$

where z, w, η, θ are the local coordinates z^1, z^2, η^1, η^2 , respectively, and $|z^i|^2 := z^i \bar{z}^i, z^i \in \{z, w\}, \eta^i \in \{\eta, \theta\}$. We choose

$$b_z = \frac{w}{|z|^2 - |w|^2}; \quad b_w = -\frac{z}{|z|^2 - |w|^2}. \quad (4.5)$$

With these tools we have constructed in [5] the complex Randers metric $\tilde{F} = \alpha + |\beta|$, where $\alpha(z, w, \eta, \theta) := \sqrt{a_{i\bar{j}}(z, w) \eta^i \bar{\eta}^j}$ and $\beta(z, \eta) = b_i(z, w) \eta^i$.

It is a complex Berwald metric, and so, by Theorem 4.3 ii), α and \tilde{F} are projectively related.

Our second goal is to find when a complex Randers metric $\tilde{F} = \alpha + |\beta|$ on a domain D from \mathbf{C}^n is projectively related to the complex Euclidean metric F on D .

For this, we make several assumptions. On the one hand we assume that \tilde{F} is a complex Berwald metric. Thus, by Theorem 4.3, ii) we obtain that α and \tilde{F} are projectively related, α is Kähler and $\tilde{G}^i = \overset{a}{G}^i$. On the other hand, we assume that α is projectively related to the Euclidean metric F . Therefore, Theorem 3.7 implies that $\overset{a}{G}^i = \frac{1}{\alpha} \frac{\partial \alpha}{\partial z^k} \eta^k \eta^i$. Under these statements, we compute

$$\begin{aligned} \frac{1}{\tilde{F}} \frac{\partial \tilde{F}}{\partial z^k} \eta^k \eta^i &= \frac{1}{\tilde{F}} \frac{\partial \alpha}{\partial z^k} \eta^k \eta^i + \frac{1}{\tilde{F}} \frac{\partial |\beta|}{\partial z^k} \eta^k \eta^i \\ &= \frac{1}{\tilde{F}} \frac{\partial \alpha}{\partial z^k} \eta^k \eta^i + \frac{1}{2|\beta|\tilde{F}} \left((\delta_k |\beta|) \eta^k + 2\bar{\beta} \overset{a}{G}^l b_l \right) \eta^i \\ &= \frac{\alpha}{\tilde{F}} \overset{a}{G}^i + \frac{|\beta|}{\tilde{F}} \frac{1}{\alpha} \frac{\partial \alpha}{\partial z^k} \eta^k \eta^i = \overset{a}{G}^i. \end{aligned}$$

Thus, $\tilde{G}^i = \frac{1}{\tilde{F}} \frac{\partial \tilde{F}}{\partial z^k} \eta^k \eta^i$, for any $i = \overline{1, n}$, which together with the Berwald assumption for \tilde{F} , give that \tilde{F} is projectively related to the complex Euclidean metric F .

Conversely, by Theorem 3.7 it results that F and \tilde{F} are projectively related if and only if the complex Randers metric \tilde{F} is weakly Kähler and $\tilde{G}^i = \frac{1}{\tilde{F}} \frac{\partial \tilde{F}}{\partial z^k} \eta^k \eta^i$, for any $i = \overline{1, n}$. These induce the generalized Berwald property for \tilde{F} and by Theorem 4.2, \tilde{F} is a complex Berwald metric. Now, taking into account Theorem 4.3, ii) it results that \tilde{F} and α are projectively related, α is Kähler and $\tilde{G}^i = \overset{a}{G}^i$.

So, we obtain

$$\overset{a}{G}^i = \frac{1}{\tilde{F}} \left(\frac{\partial \alpha}{\partial z^k} \eta^k + \frac{\bar{\beta}}{|\beta|} \overset{a}{G}^l b_l \right) \eta^i. \quad (4.6)$$

The contraction with b_i of (4.6) gives $\overset{a}{G}^i b_i = \frac{\beta}{\alpha} \frac{\partial \alpha}{\partial z^k} \eta^k$, which substituted into (4.6) yields $\overset{a}{G}^i = \frac{1}{\alpha} \frac{\partial \alpha}{\partial z^k} \eta^k \eta^i$, i.e. α is projectively related to the Euclidean metric F .

Therefore, the following theorem is proved

Theorem 4.4. *Let $\tilde{F} = \alpha + |\beta|$ be a complex Randers metric on a domain D from \mathbf{C}^n and F the complex Euclidean metric on D . Then, F and \tilde{F} are projectively related if and only if α is projectively related to the Euclidean metric F and, \tilde{F} is a complex Berwald metric.*

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References

- [1] M. Abate, G. Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.
- [2] T. Aikou, *Projective flatness of complex Finsler metrics*, Publ. Math. Debrecen **63** (2003), no. 3, 343–362.
- [3] N. Aldea, G. Munteanu, *On complex Finsler spaces with Randers metrics*, Journal Korean Math. Soc., **46** (2009), no. 5, 949-966.
- [4] N. Aldea, G. Munteanu, *On complex Landsberg and Berwald spaces*, arXiv:1010.3417v1.
- [5] N. Aldea, G. Munteanu, *On two dimensional complex Finsler manifolds*, arXiv:1010.3409v1.
- [6] P.L. Antonelli, R. Ingarden, M. Matsumoto, *The theory of sprays and Finsler spaces with applications in physics and biology*, Fundam. Th. of Ph., **58**, Kluwer Acad. Publ. Group, Dordrecht, 1993.
- [7] S. Bácsó, M. Matsumoto, *Projective changes between Finsler spaces with (α, β) -metrics*, Tensor N.S., **55** (1994), 252-257.
- [8] S. Bácsó, M. Matsumoto, *On Finsler spaces of Douglas type, A generalization of Berwald space*, Publ. Math. Debrecen, **51** (1997), 385-406.
- [9] D. Bao, S.S. Chern, Z. Shen, *An Introduction to Riemannian Finsler Geom.*, Graduate Texts in Math., **200**, Springer-Verlag, 2000.
- [10] L. Berwald, *Über Finslersche und Cartansche Geometrie. IV.*, Ann. of Math, **48** (1947), 755-781.
- [11] L. Berwald, *On Finsler and Cartan geometries, III. Two-dimensional Finsler spaces with rectilinear extremals*, Ann. of Math., (20), **42** (1941), 84-112.
- [12] T.Q. Binh, X. Cheng, *On a class of projectively flat (α, β) -Finsler metrics*, Publ. Math. Debrecen, **73** (2008), no. 3-4, 391-400.

- [13] B. Chen, Y. Shen, *Kähler Finsler metrics are actually strongly Kähler*, Chin. Ann. Math. Ser. B, **30** (2009), no. 2, 173-178.
- [14] B. Chen, Y. Shen, *On complex Randers metrics*, Int. J. of Mathematics, **21** (2010), no. 8, 971-986.
- [15] X. Cheng, Z. Shen, *Projectively flat Finsler metrics with almost isotropic S-curvature*, Acta Math. Sci. Ser. B Engl. **26** (2006), no. 2, 307-313.
- [16] S. Kobayashi, *Complex Finsler vector bundles*, Contemporary Math. 1996, 145-153.
- [17] B. Li, *On some special projectively flat (α, β) -metrics*, Publ. Math. Debrecen, **71** (2007), no. 3-4, 295-304.
- [18] M. Matsumoto, *Projective changes of Finsler metrics and projectively flat Finsler spaces*, Tensor N.S., **34** (1980), 303-315
- [19] R. Miron, M. Anastasiei, *The geometry of Lagrange Spaces; Theory and Applications*, Kluwer Acad. Publ., **59**, FTPH, 1994.
- [20] R.B. Misra, *The projective transformations in a Finsler space*, Ann. Soc. Sc. Bruxeles, **80** (1966), 755-781.
- [21] G. Munteanu, *Complex Spaces in Finsler, Lagrange and Hamilton Geometries*, Kluwer Acad. Publ., **141**, FTPH, 2004.
- [22] G. Munteanu, *Totally geodesics holomorphic subspaces*, Nonlinear Anal. Real World Appl. **8** (2007), no. 4, 1132-1143.
- [23] A. Rapcsák, *Über die Bahntreuen Abbildungen Metrischer Räume* Publ. Math. Debrecen, **8** (1961), 285-290.
- [24] H.L. Royden, *Complex Finsler metrics*, Contemporary Math., **49** (1984), 119-124.
- [25] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin, 1959.
- [26] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
- [27] Z. Szabo, *Positive definite Berwald spaces*, Tensor, N.S. **35** (1981), 25-39.

- [28] P.-Mann Wong, *A survey of complex Finsler geometry*, Advanced Stud-
ied in Pure Math., Vol. **48**, Math. Soc. of Japan, (2007), 375-433.

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